Integrability-Nonintegrability Structures and Individual Photons' Description as Finite Field Objects

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Abstract

This paper presents an attempt to come to a natural field model of individual photons considered as finite entities and propagating along some distinguished direction in space in a consistent translational-rotational manner. The starting assumption reflects their most trustful property to propagate translationally in a uniform way along straight lines. The model gives correct energy-momentum characteristics and connects the rotational characteristics of photons with corresponding nonintegrability (or curvature) of some 2-dimensional distributions (or Pfaff systems) on $\mathbb{R}^4$. It is obtained that the curvature is proportional to the corresponding energy-density. The field equations are obtained through a Lagrangian and they express a consistency condition between photon's translational and rotational propagation properties. The energy tensor is deduced directly from the equations since the corresponding Hilbert energy-tensor becomes zero on the solutions. Planck's formula $E = h\nu$ is naturally obtained as an integral translational-rotational consistency relation.

PACS:

1 Introduction

At the very dawn of the 20th century M. Planck proposed [1] and a little bit later Einstein appropriately used [2] the well known and widely used through the whole last century simple formula $E = h\nu$, $h = const > 0$. This formula marked the beginning of a new era and became a real symbol of the physical science during the following years. According to the Einstein's interpretation it gives the full energy E of *really existing* light quanta of frequency ν, and in this way a new understanding of the nature of the electromagnetic field was introduced: the field has structure which contradicts the description given by Maxwell vacuum equations. After De Broglie's suggestion for the particle-wave nature of the electron obeying the same energy-frequency relation [3], one could read Planck's formula in the following way: *there are physical objects in Nature the very existence of which is strongly connected to some periodic (with time period $T = 1/\nu$) process of intrinsic for the object nature and such that the Lorentz invariant product ET is equal to h.* Such a reading should suggest that these objects do NOT admit point-like approximation since the relativity principle for free point particles requires straight-line uniform motion, hence, no periodicity should be allowed. Instead, together with the notion

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for *finiteness* of these objects and the hypothesis for admissible point-like approximation all this led somehow to the dualistic and probabilistic viewpoint of quantum mechanics, where these (free) objects, except light quanta, have been considered as point-like but their (quantum) states have been described by (spatially infinite) cosine plane wave solutions to the potential-free Schrödinger equation [4], and the measurable physical quantities have appeared as eigen values of correspondingly defined operators satisfying definite commutation relations.

Although the great (from pragmatic point of view) achievements of such an approach, known as *quantum theory*, the great challenge to build an adequate description of individual representatives of these objects, especially of light quanta called by Lewis *photons* [5], is still to be appropriately met since the efforts made in this direction, we have to admit, still have not brought satisfactory results. Recall that Einstein in his late years recognizes [6] that "the whole fifty years of conscious brooding have not brought me nearer to the answer to the question "what are light quanta", and now, half a century later, theoretical physics, although some existing attempts [7-19], still needs progress to present a satisfactory answer to the question "what is a photon". We consider the corresponding theoretically directed efforts as necessary and even *urgent* in view of the growing amount of definite experimantal skills in manipulation with individual photons, in particular, in connection with the experimental advancement in the "quantum computer" project. The dominating modern theoretical view on microobjects is based on the notions and concepts of quantum field theory (QFT) where the structure of the photon (as well as of any other microobject) is accounted for mainly through the so called *structural function*, and highly expensive and delicate collision experiments are planned and carried out namely in the frame of these concepts and methods (see the 'PHOTON' Conferences Proceedings, some recent review papers: [20-23]). Going not in details we just note a special feature of this QFT approach: if the study of a microobject leads to conclusion that it has structure, i.e. it is not point-like, then the corresponding constituents of this structure are considered as point-like, so the point-likeness stays in the theory just in a lower level.

In this paper we follow another approach based on the assumption that the description of the available (most probaly NOT arbitrary) spatial structure of photons can be made by *continuous finite/localized* functions of the three space variables. The difficulties met in this approach consist mainly, in our view, in finding adequate enough mathematical objects and solving appropriate PDE. The lack of sufficiently reliable corresponding information made us look into the problem from as general as possible point of view on the basis of those properties of photons which may be considered as most undoubtedly trustful. The analysis made suggested that such a property seems to be the fact that *the propagation of an individual photon necessarily includes a straight-line uniform component*, so we shall focus on this property in order to see what useful for our purpose suggestions could be deduced and what appropriate structures could be constructed. All these suggestions and structures should be the building material for a step-by-step creating a *self-consistent* system, i.e. the corresponding properties must live in harmony with each other. From theoretical viewpoint, these properties should reflect various important aspects of some general notion about individual photons, and we now turn to outline briefly such a notion.

2 The Notion

We introduce the following concept of individual photons, and we consider it as working one, so it may be subject to future improvements.

Photons are real massless time-stable physical objects with a consistent translational-rotational dynamical structure.

We give now some explanatory comments, beginning with the term *real*. **First** we emphasize that this term means that we consider photons as *really* existing *physical* objects, not as

appropriate and helpful but imaginary (theoretical) entities. Accordingly, photons **necessarily carry energy-momentum**, otherwise, they could hardly be detectable. **Second**, photons can undoubtedly be *created* and *destroyed*, so, no point-like and infinite models are reasonable: point-like objects are assumed to have no structure, so they can not be destroyed since there is no available structure to be destroyed; creation of infinite objects (e.g. plane waves) requires infinite quantity of energy to be transformed from one kind to another for finite time-period, which seems also unreasonable. Accordingly, photons are *spatially finite* and have to be modeled like such ones, which is the only possibility to be consistent with their "created-destroyed" nature. It seems hardly believable that spatially infinite and indestructible physical objects may exist at all. **Third**, "spatially finite" implies that photons may carry only *finite values* of physical (conservative or non-conservative) quantities. In particular, the most universal physical quantity seems to be the energy-momentum, so the model must allow finite integral values of energy-momentum to be carried by the corresponding solutions. **Fourth**, "spatially finite" means also that photons *propagate*, i.e. they do not "move" like classical particles along trajectories, therefore, partial differential equations should be used to describe their evolution in time.

The term "**massless**" characterizes the way of propagation: the integral energy E and momentum p of a photon should satisfy the relation $E = cp$, where c is the speed of light in vacuum, and in relativistic terms this means that their integral energy-momentum vector *must be isotropic*, i.e. it must have zero module with respect to Lorentz-Minkowski (psuedo)metric in $\mathbb{R}^4$. Since the translational velocity of every point where the corresponding field functions are different from zero must be equal to c, we have in fact an isotropic vector field $\bar{\zeta}$. The integral trajectories of this vector field are isotropic straight lines. It follows that just the corresponding direction is important, so, canonical coordinates $(x^1, x^2, x^3, x^4) = (x, y, z, \xi = ct)$ on $\mathbb{R}^4$ may be chosen such that $\bar{\zeta}$ may have only two non-zero components of magnitude 1: $\bar{\zeta}^\mu = (0, 0, -\varepsilon, 1)$, where $\varepsilon = \pm 1$ accounts for the two directions along the cooedinate z. Further such a coordinate system will be called $\bar{\zeta}$-adapted and will be of main usage. It may be also expectable, that the corresponding energy-momentum tensor $T_{\mu\nu}$ of the model satisfies the relation $T_{\mu\nu}T^{\mu\nu} = 0$, which may be considered as a localization of the integral isotropy condition $E^2 - c^2p^2 = 0$.

The term "**translational-rotational**" means that besides translational component along $\bar{\zeta}$, the propagation demonstrates also some rotational (in the general sense of this concept) component in such a way that *both components exist simultaneously and consistently*. It seems reasonable to expect that such kind of behavior should be consistent only with some distinguished spatial shapes. Moreover, if the Planck relation $E = h\nu$ must be respected throughout the evolution, the rotational component of propagation should have *periodical* nature and one of the two possible, *left* or *right*, orientations. It seems reasonable also to expect periodicity in the spatial shape of photons, which somehow to be related to the time periodicity.

The term "**dynamical structure**" means that the propagation is supposed to be necessarily accompanied with *internal energy-momentum redistribution*, which may be considered in the model as energy-momentum exchange between (or among) somewhat functionally isolated subsystems. It could also mean that photons live in a dynamical harmony with the outside world, i.e. *any outside directed energy-momentum flow should be accompanied by a parallel inside directed energy-momentum flow.* From theoretical point of view this would mean that the corresponding mathematical model-object should be a many-component one, and most probably, the components should be vectors, tensors, etc. In other words, the mathematical model-object may have two or more independent systems of indices.

Finally, note that if the time periodicity and the spatial periodicity should be consistent somehow, the simplest such consistency would seem like this: the spatial longitudinal dimension $2\pi l_o$ is equal to cT: $2\pi l_o = cT$, where l_o is some finite positive characteristic constant of the corresponding solution. This would mean that every individual photon determines its own

length/time scale.

From differential viewpoint the translational-rotational consistency should mean that every translational change (i.e. change along $\bar{\zeta}$) of the mathematical object F representing an individual photon, which change is naturally given by the Lie derivative $L_{\bar{\zeta}}F$ of F along $\bar{\zeta}$, must be accompanied by a "rotational" change $D(F)$ of F, where D is some rotation generating operator. So, we should expect the appearance of a relation stating some simple and strong dependence (e.g. proportionality) between $\kappa L_{\bar{\zeta}}F$ and $D(F)$, where $\kappa = \pm 1$ accounts for the *left/right* rotational orientations.

According to the above $\bar{\zeta}$ turns out to be the most natural and trustful mathematical object that has local nature and that could be used to induce other appropriately consistent and physically useful local geometric structures on $\mathbb{R}^4$. So, in the next section we are going to study some of these structures.

3 Some Geometry on $\mathbb{R}^4$ induced by $\bar{\zeta}$

We consider the space $\mathbb{R}^4$ as a manifold related to standard global coordinates

$$(x^1, x^2, x^3, x^4) = (x, y, z, \xi = ct)$$

and shall not make use of the Minkowski metric for the time being. The only structures on $\mathbb{R}^4$ assumed to exist for now are the vector field $\bar{\zeta}$ and the natural volume form $\omega_o = dx\wedge dy\wedge dz\wedge d\xi$. We choose the coordinates in such a way that $\bar{\zeta}$ looks as follows:

$$\bar{\zeta} = -\varepsilon\frac{\partial}{\partial z} + \frac{\partial}{\partial \xi}, \quad \varepsilon = \pm 1. \tag{1}$$

Since every vector field on a manifold generates 1-dimensional locally integrable differential system the same should be true for our field $\bar{\zeta}$. Let's denote the corresponding 3-dimensional Pfaff system by $\Delta^*(\bar{\zeta})$. Thus, $\Delta^*(\bar{\zeta})$ is generated by three linearly independent 1-forms $(\alpha_1, \alpha_2, \alpha_3)$ which annihilate $\bar{\zeta}$, i.e.

$$\alpha_1(\bar{\zeta}) = \alpha_2(\bar{\zeta}) = \alpha_3(\bar{\zeta}) = 0; \quad \alpha_1\wedge\alpha_2\wedge\alpha_3 \neq 0.$$

The following basis of $\Delta^*(\bar{\zeta})$ will be of use, and instead of $(\alpha_1, \alpha_2, \alpha_3)$ we introduce the notation (A, A^*, ζ):

$$A = udx + pdy; \quad A^* = -pdx + udy; \quad \zeta = \varepsilon dz + d\xi, \tag{2}$$

where (u, p) are two arbitrary functions.

Proposition 1. The following relations hold:

$$\mathbf{d}A\wedge A\wedge A^* = 0; \quad \mathbf{d}A^*\wedge A^*\wedge A = 0; \tag{3}$$

$$\mathbf{d}A\wedge A\wedge\zeta = \varepsilon\big[u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)\big]\omega_o; \quad \mathbf{d}A^*\wedge A^*\wedge\zeta = \varepsilon\big[u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)\big]\omega_o. \tag{4}$$

These relations (3)-(4) say that the 2-dimensional Pfaff system (A, A^*) is completely integrable for any choice of the two functions (u, p), while the two 2-dimensional Pfaff systems (A, ζ) and (A^*, ζ) are NOT completely integrable in general, and the same curvature factor $\mathbf{R} = u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)$ determines their nonintegrability.

In order to give the dual non-integrability picture we consider the 1-form ζ. It determines a 3-dimensional distribution (or differential system) $\Delta(\zeta)$, and a basis of this distribution is given by the vector fields $(\bar{A}, \bar{A}^*, \bar{\zeta})$, where

$$\bar{A} = -u\frac{\partial}{\partial x} - p\frac{\partial}{\partial y}; \quad \bar{A}^* = p\frac{\partial}{\partial x} - u\frac{\partial}{\partial y}; \tag{5}$$

and the vector field $\bar{\zeta}$ is given by (1): $\zeta(\bar{A}) = \zeta(\bar{A}^*) = \zeta(\bar{\zeta}) = 0$; $\bar{A}\wedge\bar{A}^*\wedge\bar{\zeta} \neq 0$.

Further in the paper we shall relate the 2-forms on $\mathbb{R}^4$ to the basis

$$dx\wedge dy, \quad dx\wedge dz, \quad dy\wedge dz, \quad dx\wedge d\xi, \quad dy\wedge d\xi, \quad dz\wedge d\xi.$$

It deserves to note here that one of the canonical complex structures in the bundle of 2-forms, denoted by $\mathcal{J}$, and given by

$$\mathcal{J}(dx\wedge dy) = -dz\wedge d\xi, \quad \mathcal{J}(dx\wedge dz) = dy\wedge d\xi, \quad \mathcal{J}(dy\wedge dz) = -dx\wedge d\xi,$$

$$\mathcal{J}(dx\wedge d\xi) = dy\wedge dz, \quad \mathcal{J}(dy\wedge d\xi) = -dx\wedge dz, \quad \mathcal{J}(dz\wedge d\xi) = dx\wedge dy$$

transforms $A\wedge\zeta$ to $A^*\wedge\zeta$.

Proposition 2. The following Lie-bracket relations hold:

$$[\bar{A},\bar{\zeta}] = (u_\xi - \varepsilon u_z)\frac{\partial}{\partial x} + (p_\xi - \varepsilon p_z)\frac{\partial}{\partial y}; \tag{6}$$

$$[\bar{A}^*,\bar{\zeta}] = -(p_\xi - \varepsilon p_z)\frac{\partial}{\partial x} + (u_\xi - \varepsilon u_z)\frac{\partial}{\partial y}. \tag{7}$$

In accordance with **Prop.1** from these last relations (6)-(7) it follows that the two distributions $(\bar{A},\bar{\zeta})$ and $(\bar{A}^*,\bar{\zeta})$ would be completely integrable only if the same curvature factor

$$\mathbf{R} = u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z) \tag{8}$$

is zero. In fact, for example, if $(\bar{A},\bar{\zeta})$ is completely integrable then there must exist two functions f and g such that the Lie bracket $[\bar{A},\bar{\zeta}]$ must be representable as follows:

$$[\bar{A},\bar{\zeta}] = f\bar{A} + g\bar{\zeta}.$$

Since $[\bar{A},\bar{\zeta}]$ has no components along $\frac{\partial}{\partial z}$ and $\frac{\partial}{\partial \xi}$ the above relation reduces to

$$(u_\xi - \varepsilon u_z)\frac{\partial}{\partial x} + (p_\xi - \varepsilon p_z)\frac{\partial}{\partial y} = f\bar{A} = -fu\frac{\partial}{\partial x} - fp\frac{\partial}{\partial y}.$$

From this last relation it follows

$$-f = \frac{u_\xi - \varepsilon u_z}{u} = \frac{p_\xi - \varepsilon p_z}{p},$$

i.e. $u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z) = \mathbf{R} = 0$.

We mention also that the projections

$$\langle A, [\bar{A}^*,\bar{\zeta}]\rangle = -\langle A^*, [\bar{A},\bar{\zeta}]\rangle = u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z)$$

give the same factor $\mathbf{R}$. The same curvature factor appears, of course, as coefficient in the exterior products

$$[\bar{A}^*,\bar{\zeta}]\wedge\bar{A}^*\wedge\bar{\zeta} \quad \text{and} \quad [\bar{A},\bar{\zeta}]\wedge\bar{A}\wedge\bar{\zeta}.$$

For the other two projections we obtain

$$\langle A, [\bar{A},\bar{\zeta}]\rangle = -\langle A^*, [\bar{A}^*,\bar{\zeta}]\rangle = \frac{1}{2}\big[(u^2+p^2)_\xi - \varepsilon(u^2+p^2)_z\big]. \tag{9}$$

Clearly, the last relation (9) may be put in terms of the Lie-derivative as

$$\frac{1}{2}L_{\bar{\zeta}}(u^2+p^2) = -\frac{1}{2}L_{\bar{\zeta}}\langle A,\bar{A}\rangle = -\langle A, L_{\bar{\zeta}}\bar{A}\rangle = -\langle A^*, L_{\bar{\zeta}}\bar{A}^*\rangle.$$

Remark. Further in the paper we shall denote $\sqrt{u^2+p^2}\equiv\phi$, and shall assume that ϕ is a *spatially finite* function, so, u and p must also be spatially finite.

It deserves to note that the tensor field $T=\phi^2\zeta\otimes\bar\zeta$, considered as a linear map, satisfies the relation $T\circ T=0$, so it is a boundary operator with image space generated by $\bar\zeta$ and kernel space generated by the vectors $(\bar A,\bar A^*,\bar\zeta)$, so, the corresponding homology space is generated by the classes of $\bar A$ and $\bar A^*$. Moreover, in this coordinate system we have $T^4_4=\phi^2$ and $tr(T\circ T)=T^\nu_\mu T^\mu_\nu=0$, so T seems to be an appropriate candidate for energy-momentum tensor of a photon-like solution. As it is easily seen a pseudoeuclidean metric η in the tangent bundle generates a euclidean metric g on this homology space such that on the classes $[\bar A]$ and $[\bar A^*]$ we have $||[\bar A]||=\sqrt{|\eta(\bar A,\bar A)|}$, and $||[\bar A^*]||=\sqrt{|\eta(\bar A^*,\bar A^*)|}$.

Proposition 3. There is a function $\psi(u,p)$ such, that

$$L_{\bar\zeta}\psi=\frac{u(p_\xi-\varepsilon p_z)-p(u_\xi-\varepsilon u_z)}{\phi^2}=\frac{\mathbf{R}}{\phi^2}.$$

Proof. It is immediately checked that $\psi=\arctan\frac{p}{u}$ is such one.

We note that the function ψ has a natural interpretation of *phase* because of the easily verified now relations

$$u=\phi\cos\psi,\quad p=\phi\sin\psi,$$

and ϕ acquires the status of *amplitude.* Since the transformation $(u,p)\rightarrow(\phi,\psi)$ is non-degenerate this allows to work with the two functions (ϕ,ψ) instead of (u,p).

Note that the amplitude ϕ appears as square root of the determinant function of the basis transformation

$$\left\{\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z},\frac{\partial}{\partial\xi}\right\}\rightarrow\left\{\bar A,\bar A^*,\frac{\partial}{\partial z},\frac{\partial}{\partial\xi}\right\},$$

and the function ψ appears as $arc\cos(1-\frac{1}{2}\theta)$ where θ is the trace of the basis transformation

$$\left\{\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z},\frac{\partial}{\partial\xi}\right\}\rightarrow\left\{\frac{\bar A}{\phi},\frac{\bar A^*}{\phi},\frac{\partial}{\partial z},\frac{\partial}{\partial\xi}\right\}.$$

It follows that ϕ and ψ have well defined invariant sense.

Finally we note that the same curvature factor may be obtained through declaring the subspaces generated by $(\bar A,\bar\zeta)$ and $(\bar A^*,\bar\zeta)$ as *horizontal*, and the subspaces generated respectively by $\left(\bar A^*,\varepsilon\frac{\partial}{\partial z}+\frac{\partial}{\partial\xi}\right)$ and $\left(\bar A,\varepsilon\frac{\partial}{\partial z}+\frac{\partial}{\partial\xi}\right)$ as *vertical*, and computing the corresponding vertical projections, respectively, of $[\bar A,\bar\zeta]$ and $[\bar A^*,\bar\zeta]$.

From **Prop.3** we have

$$\mathbf{R}=\phi^2L_{\bar\zeta}\psi=\ \phi^2(\psi_\xi-\varepsilon\psi_z)\,. \tag{10}$$

This last formula (10) shows something very important: at any $\phi\neq0$ the curvature $\mathbf{R}$ will NOT be zero only if the phase ψ is NOT a running wave along $\bar\zeta$, i.e. only if $L_{\bar\zeta}\psi\neq0$, which admits in principle availability of rotation. In fact, lack of rotation would mean that ϕ and ψ are running waves along $\bar\zeta$. The relation $L_{\bar\zeta}\psi\neq0$ means, however, that rotational properties are possible in general, and some of these properties are carried by the phase ψ. It follows that in such a case the translational component of propagation along $\bar\zeta$ (which is supposed to be available) must be determined essentially, and most probably entirely, by ϕ. In particular, we could expect the relation $L_{\bar\zeta}\phi=0$ to hold, and if this happens, then the rotational component of propagation will be represented entirely by the phase ψ, and, more specially, by the curvature factor $\mathbf{R}\neq0$, so, further we assume that, in general, $L_{\bar\zeta}\psi\neq0$.

We consider now the Lie-brackets $[\bar{A},\bar{\zeta}]$ and $[\bar{A}^*,\bar{\zeta}]$ in terms of ψ:

$$[\bar{A},\bar{\zeta}]=-L_{\bar{\zeta}}\psi.\bar{A}^* \ ; \quad [\bar{A}^*,\bar{\zeta}]=L_{\bar{\zeta}}\psi.\bar{A} \ ; \quad [\bar{A},\bar{\zeta}]\wedge[\bar{A}^*,\bar{\zeta}]=-\left(L_{\bar{\zeta}}\psi\right)^2 \bar{A}^*\wedge\bar{A}\neq 0.$$

These relations say that the 2-dimensional frame $(\bar{A},\bar{A}^*)$ on the (x,y)-plane is transformed to the 2-dimensional frame $([\bar{A},\bar{\zeta}],[\bar{A}^*,\bar{\zeta}])$ on the same plane by means of the matrix $L_{\bar{\zeta}}\psi.J$, where J is the canonical complex structure in $\mathbb{R}^2$, generating rotation to the angle of $\pi/2$. Hence, the 4-frame $(\bar{A},\bar{A}^*,\partial_z,\partial_\xi)$ is transformed to the 4-frame $([\bar{A},\bar{\zeta}],[\bar{A}^*,\bar{\zeta}],\partial_z,\partial_\xi)$ by means of a linear map with determinant $(L_{\bar{\zeta}}\psi)^2$. This observation confirms once again that at $\phi\neq 0$ **the rotational component of propagation will be available only if $L_{\bar{\zeta}}\psi\neq 0$, i.e. only if the non-integrability factor R is non-zero**. Moreover, it is suggested to make use of some complex structures as possible rotation-generating operators.

The two nonintegrable Pfaff systems (A,ζ) and (A^*,ζ) define corresponding 2-forms: $G=A\wedge\zeta$ and $G^*=A^*\wedge\zeta$. Let's denote $\bar{A}\wedge\bar{\zeta}\equiv\bar{G}$, $\bar{A}^*\wedge\bar{\zeta}\equiv\bar{G}^*$. We shall need the quantities $i(\bar{G})\mathbf{d}G+i(\bar{G}^*)\mathbf{d}G^*$ and $i(\bar{G}^*)\mathbf{d}G+i(\bar{G})\mathbf{d}G^*$, where $i(\bar{G})=i(\bar{\zeta})\circ i(\bar{A})$, and, analogically, $i(\bar{G}^*)=i(\bar{\zeta})\circ i(\bar{A}^*)$, and $i(X)$ is the standard insertion operator in the exterior algebra of differential forms on $\mathbb{R}^4$ defined by the vector field X.

Having in view the explicit expressions for $A,A^*,\zeta,\bar{A},\bar{A}^*$ and $\bar{\zeta}$, and making use of the coordinate free formula for the exterior derivative of 1-form α:

$$\mathbf{d}\alpha(X,Y)=X\langle\alpha,Y\rangle-Y\langle\alpha,X\rangle-\alpha([X,Y])$$

we obtain

$$i(\bar{G})\mathbf{d}G=i(\bar{G}^*)\mathbf{d}G^*=\frac{1}{2}L_{\bar{\zeta}}\left(\phi^2\right).\zeta\ ,$$

$$i(\bar{G})\mathbf{d}G+i(\bar{G}^*)\mathbf{d}G^*=\frac{1}{2}\Big[(\bar{G})^{\alpha\beta}(\mathbf{d}G)_{\alpha\beta\mu}+(\bar{G}^*)^{\alpha\beta}(\mathbf{d}G^*)_{\alpha\beta\mu}\Big]dx^\mu=L_{\bar{\zeta}}\left(\phi^2\right).\zeta\ . \qquad (11)$$

Also, we have

$$i(\bar{G}^*)\mathbf{d}G=-i(\bar{G})\mathbf{d}G^*\ ,$$

$$i(\bar{G}^*)\mathbf{d}G+i(\bar{G})\mathbf{d}G^*=\frac{1}{2}\Big[(\bar{G}^*)^{\alpha\beta}(\mathbf{d}G)_{\alpha\beta\mu}+(\bar{G})^{\alpha\beta}(\mathbf{d}G^*)_{\alpha\beta\mu}\Big]dx^\mu=0\ . \qquad (12)$$

A direct calculation shows that

$$i(\bar{G}^*)\mathbf{d}G=-i(\bar{G})\mathbf{d}G^*=$$

$$=\varepsilon\Big[u(p_\xi-\varepsilon p_z)-p(u_\xi-\varepsilon u_z)\Big]dz+\Big[u(p_\xi-\varepsilon p_z)-p(u_\xi-\varepsilon u_z)\Big]d\xi=\mathbf{R}.\zeta\ .$$

We note that if η is the pseudoeuclidean metric with signature $(-,-,-,+)$, then making use of the corresponding Hodge $*$-operator, the following relations can be easily verified:

$$(\varepsilon G^*)_{\mu\nu}=(*G)_{\mu\nu},\ \bar{A}^\mu=\eta^{\mu\nu}A_\nu,\ \bar{\zeta}^\mu=\eta^{\mu\nu}\zeta_\nu,\ \ \bar{G}^{\mu\nu}=\eta^{\mu\sigma}\eta^{\nu\tau}G_{\sigma\tau},\ \ (\varepsilon\bar{G}^*)^{\mu\nu}=\eta^{\mu\sigma}\eta^{\nu\tau}(*G)_{\sigma\tau}\ .$$

Moreover, with respect to the corresponding Levi-Civita covariant derivative ∇ we obtain

$$\nabla_\nu\left[\frac{1}{4}G_{\alpha\beta}\bar{G}^{\alpha\beta}\delta^\nu_\mu-G_{\mu\sigma}\bar{G}^{\nu\sigma}\right]=\frac{1}{2}\Big[(\bar{G})^{\alpha\beta}(\mathbf{d}G)_{\alpha\beta\mu}+(*\bar{G})^{\alpha\beta}(\mathbf{d}*G)_{\alpha\beta\mu}\Big]\ . \qquad (13)$$

Relations (11) and (13) show that if $L_{\bar{\zeta}}\phi=0$, then the well known Maxwell energy-momentum tensor, determined by the 2-form $G=A\wedge\zeta$ has zero divergence, and additionally, we get

$$\phi^2\zeta_\mu\bar{\zeta}^\nu=\frac{1}{4}G_{\alpha\beta}\bar{G}^{\alpha\beta}\delta^\nu_\mu-G_{\mu\sigma}\bar{G}^{\nu\sigma}=\frac{1}{2}\Big[G_{\mu\sigma}\bar{G}^{\nu\sigma}+(*G)_{\mu\sigma}(*\bar{G})^{\nu\sigma}\Big]\ . \qquad (14)$$

We consider now a consequence of having a completely integrable 1-dimensional Pfaff system on $\mathbb{R}^4$ defined by the 1-form $\omega : \mathbf{d}\omega \wedge \omega = 0$. This 1-dimensional Pfaff system is determined up to a nonvanishing function f: $\omega \rightarrow f\omega, f(x) \neq 0, x \in M$, since $f\omega$ satisfies the same equation $\mathbf{d}(f\omega) \wedge (f\omega) = 0$. It follows that there is 1-form θ (in fact a class of 1-forms $(\theta + g\omega)$, g is a function) such, that $\mathbf{d}\omega = \theta \wedge \omega$. Now, the Godbillon-Vey theorem says that the 3-form $\beta = \mathbf{d}\theta \wedge \theta$ is closed: $\mathbf{d}\beta = \mathbf{d}(\mathbf{d}\theta \wedge \theta) = 0$, and, varying θ and ω in an admissible way: $\theta \rightarrow (\theta + g\omega)$; $\omega \rightarrow f\omega$, leads to adding an exact 3-form to β, so we have a cohomological class Γ defined entirely by the integrable 1-dimensional Pfaff system. From physical point of view this seems to be important because it shows that each completely integrable 1-dimensional Pfaff system on Minkowski space may generate a conservation law through integrating over $\mathbb{R}^3$ the restriction of β to $\mathbb{R}^3$, provided this integral is finite.

Recall now our objects: A, A^*, ζ and the corresponding vector fields $\bar{A}, \bar{A}^*, \bar{\zeta}$. Let's consider the 1-form $\omega = f\zeta = \varepsilon f dz + f d\xi$, where f is a nonvanishing function on M. We have the relations:

$$\omega(\bar{A}) = 0, \quad \omega(\bar{A}^*) = 0, \quad \omega(\bar{\zeta}) = 0.$$

Moreover, since ζ is closed, $\omega = f\zeta$ satisfies the Frobenius integrability condition:

$$\mathbf{d}\omega \wedge \omega = f\mathbf{d}f \wedge \zeta \wedge \zeta = 0.$$

Therefore, the corresponding 1-dimensional Pfaff system, defined by $\omega = f\zeta$, is completely integrable, and there exists a new 1-form θ, such that $\mathbf{d}\omega = \theta \wedge \omega$, and $\mathbf{d}(\mathbf{d}\theta \wedge \theta) = 0$.

Clearly, the natural imbedding $i : (x, y, z) \rightarrow (x, y, z, 0)$ will give the natural restriction $i^*\beta$ to $\mathbb{R}^3$. We note that from our point of view it is not so important whether Γ is trivial or nontrivial. The important point is the 3-form β to have appropriate component β_{123} in front of the basis element $dx \wedge dy \wedge dz$, because only this component survives after the restriction considered is performed, which formally means that we put $\xi = 0$ and $d\xi = 0$ in β. The value of the corresponding conservative quantity will be found provided the integration can be carried out successfully, i.e. when $(i^*\beta)_{123}$ is concentrated in a finite 3d-subregion of $\mathbb{R}^3$ with no singularities.

In order to find appropriate θ in our case we are going to take advantage of the freedom we have when choosing θ: the 1-form θ is defined up to adding to it an 1-form $\alpha = g\,\omega$, where g is an arbitrary function on M, because θ is defined by the relation $\mathbf{d}\omega = \theta \wedge \omega$, and $(\theta + g\,\omega) \wedge \omega = \theta \wedge \omega$ always. The freedom in choosing ω consists in choosing the function f, and we shall show that f may be chosen in such a way: $\omega = f\,\zeta$, that the corresponding integral of $i^*\beta$ to present a finite conservative quantity.

Recalling that $\mathbf{d}\zeta = 0$, we have

$$\mathbf{d}\omega = \mathbf{d}(f\zeta) = \mathbf{d}f \wedge \zeta + f\mathbf{d}\zeta = \mathbf{d}f \wedge \zeta.$$

Since $\mathbf{d}\omega$ must be equal to $\theta \wedge \omega$ we obtain

$$\mathbf{d}\omega = \mathbf{d}f \wedge \zeta = \theta \wedge \omega = \theta \wedge (f\zeta) = f\theta \wedge \zeta.$$

It follows

$$\theta \wedge \zeta = \frac{1}{f}\mathbf{d}f \wedge \zeta = \mathbf{d}(ln\, f) \wedge \zeta = \Big[\mathbf{d}(ln\, f) + h\zeta\Big] \wedge \zeta,$$

where h is an arbitrary function. Hence, in general, we obtain $\theta = \mathbf{d}(ln\, f) + h\zeta$. Therefore, $\mathbf{d}\theta = \mathbf{d}h \wedge \zeta$ and for $\beta = \mathbf{d}\theta \wedge \theta$ we obtain

$$\beta = \mathbf{d}\theta \wedge \theta = (\mathbf{d}h \wedge \zeta) \wedge \big[\mathbf{d}(ln\, f) + h\zeta\big] = \big[\mathbf{d}(ln\, f)\big] \wedge \mathbf{d}h \wedge \zeta.$$

Denoting for convenience $(ln\, f) = \varphi$ for the restriction $i^*\beta$ we obtain

$$i^*\beta = \varepsilon(\varphi_x h_y - \varphi_y h_x) dx \wedge dy \wedge dz.$$

In order to find appropriate interpretation of $i^*\beta$ we easily check that the 3-form

$$i(\bar{\zeta})(\mathbf{d}A \wedge A \wedge \zeta) = i(\bar{\zeta})(\mathbf{d}A^* \wedge A^* \wedge \zeta) \tag{15}$$

looks like $\gamma \wedge \zeta$ with $\gamma = -\phi^2(\psi_\xi - \varepsilon\psi_z) dx \wedge dy$. Moreover, if

$$\phi^2 = \phi^2(x, y, \xi + \varepsilon z) \ \text{ and } \ (\psi_\xi - \varepsilon\psi_z) = \pm\frac{1}{l_o} = const$$

this 3-form is closed: $\mathbf{d}(\gamma \wedge \zeta) = 0$.

Hence, the interpretation $i^*\beta = i^*(\gamma \wedge \zeta)$ requires appropriate definition of the two functions f and h, i.e. we must have

$$\varphi_x h_y - h_x \varphi_y = -\frac{\kappa}{l_o}\phi^2, \quad \kappa = \pm 1.$$

If we choose

$$f = exp(\varphi) = exp\left[\int \phi^2 dx\right], \quad h = -\frac{\kappa}{l_o} y + const$$

all requirements will be fulfilled, in particular, $\varphi_{xy} = \varphi_{yx} = (\phi^2)_y$ and $h_{xy} = h_{yx} = 0$.

The above choice of f and h allows the spatial restriction of the Godbillon-Vey 3-form β to be physically interpreted. So, the curvature expressions and the rotational properties of the corresponding solutions can be related to the integrability of the Pfaff system defined by ζ, i.e. to the straight-line translational propagation properties of the solutions. Hence, a consistent translational-rotational propagation is possible.

4 Lagrangian formulation

First we consider a simpler but helpful and suggestive example of a lagrangian defined by a complex valued function. Let the field of complex numbers $\mathbb{C} = (\mathbb{R}^2, J), J \circ J = -id_{\mathbb{R}^2}$ be given a real representation as a 2-dimensional real vector space with basis

$$I = \begin{Vmatrix} 1 & 0 \\ 0 & 1 \end{Vmatrix}, \quad J = \begin{Vmatrix} 0 & 1 \\ -1 & 0 \end{Vmatrix}.$$

Note that the complex structure operator J is a rotational operator inducing rotation in $\mathbb{R}^2$ to the angle of $\pi/2$. Every $\mathbb{C}$-valued function α on $\mathbb{R}^4$ can be represented in the form $\alpha = uI + pJ = \phi\cos\psi\, I + \phi\sin\psi\, J$, where u and p are two real-valued functions, $\phi = \sqrt{u^2 + p^2}$ and $\psi = \arctan\frac{p}{u}$, and the components of α with respect to this basis will be anumerated by latin indices taking values $(1,2) : \alpha_i, i = 1,2$. We denote further $J(\alpha) = -pI + uJ \equiv \bar{\alpha}$. We introduce the two 1-forms (in our ζ-adapted coordinate system): $k^s = i^*\zeta/l_o + 0.d\xi$ and $k^\xi = \xi^*\zeta/l_o + 0.dx + 0.dy + 0.dz$ - the l_o-scaled ξ-restriction of ζ, with zero elements along the spatial directions, $l_o = const > 0$.

Making use of our vector field $\bar{\zeta}$, of the inner product g in $(\mathbb{R}^2, J)$ defined by $g(\alpha, \beta) = \frac{1}{2} tr(\alpha \circ \beta^*)$ and working in the corresponding $\bar{\zeta}$-adapted coordinate system (x, y, z, ξ), we consider now the following lagrangian (summation over the repeated indices: $g(\alpha, \alpha) = \alpha_i \alpha_i$):

$$\mathbb{L} = \frac{1}{2}\Big[\kappa l_o g(\alpha, L_{\bar{\zeta}}\bar{\alpha}) + g(\alpha, \alpha) - \kappa l_o g(\bar{\alpha}, L_{\bar{\zeta}}\alpha) + g(\bar{\alpha}, \bar{\alpha})\Big] =$$

$$= \frac{1}{2}\Big[\alpha_i\Big(\kappa l_o \bar{\zeta}^\sigma \frac{\partial \bar{\alpha}_i}{\partial x^\sigma} + \alpha_i\Big) - \bar{\alpha}_i\Big(\kappa l_o \bar{\zeta}^\sigma \frac{\partial \alpha_i}{\partial x^\sigma} - \bar{\alpha}_i\Big)\Big],$$

where l_o is of dimension [length], and $\kappa = \pm 1$. The Lagrange equations are

$$\kappa l_o L_{\bar{\zeta}} \bar{\alpha} = -\alpha \ ; \quad \kappa l_o L_{\bar{\zeta}} \alpha = \bar{\alpha},$$

or, in components,

$$\kappa l_o \bar{\zeta}^\sigma \frac{\partial \bar{\alpha}_i}{\partial x^\sigma} = -\alpha_i \ ; \quad \kappa l_o \bar{\zeta}^\sigma \frac{\partial \alpha_i}{\partial x^\sigma} = \bar{\alpha}_i.$$

Note that the first equation follows from the second one under the acting with J from the left, hence, we have just one equation of the form $\kappa l_o L_{\bar{\zeta}} \alpha = J(\alpha)$, which represents, as we mentioned earlier, the idea for *consistent translational-rotational propagation.* Moreover, since J rotates to the angle of $\pi/2$, the parameter l_o fixes the corresponding translational advancement.

In terms of (ϕ, ψ) these equations give

$$L_{\bar{\zeta}}\phi.\cos(\psi) - \phi.\sin(\psi)\left(L_{\bar{\zeta}}\psi - \frac{\varepsilon\kappa}{l_o}\right) = 0, \quad L_{\bar{\zeta}}\phi.\sin(\psi) + \phi.\cos(\psi)\left(L_{\bar{\zeta}}\psi - \frac{\varepsilon\kappa}{l_o}\right) = 0,$$

where $\phi = \sqrt{\alpha_1^2 + \alpha_2^2}$ and $\psi = \arctan \frac{\alpha_2}{\alpha_1}$. These two equations are consistent only if

$$L_{\bar{\zeta}}\phi = 0, \quad L_{\bar{\zeta}}\psi = \varepsilon \frac{\kappa}{l_o}. \tag{16}$$

So, equations (16) are our dynamical equations, and their solutions are:

$$\phi = \phi(x, y, \xi + \varepsilon z); \quad \psi_1 = \psi_o(x, y, \xi + \varepsilon z) - \frac{\kappa}{l_o} z + C_1; \quad \psi_2 = \psi_o(x, y, \xi + \varepsilon z) + \frac{\kappa\varepsilon}{l_o}\xi + C_2, \tag{17}$$

where C_1 and C_2 may depend on (x, y). Omitting the running wave term ψ_o and assuming $C_1 = const$ and $C_2 = const$ we see that

$$\psi_1 = -\frac{\kappa}{l_o} z + const = -k^s_\mu x^\mu + const \quad \text{and} \quad \psi_2 = \frac{\kappa\varepsilon}{l_o}\xi + const = k^\xi_\mu x^\mu + const$$

are the essential possible solutions leading to non-zero curvature and, therefore, to available rotational component of propagation.

It is important to note that this lagrangian leads to *linear* equations for the components of α, which equations admit 3d-finite solutions of the kind

$$\alpha_1 = \phi \cos\psi; \quad \alpha_2 = \phi \sin\psi$$

with consistent translational-rotational behavior, where ϕ and ψ are given above, and ϕ is a spatially finite function.

It is easily seen that the lagrangian becomes zero on the solutions, and since this lagrangian does NOT depend on any space-time metric the corresponding Hilbert energy-momentum tensor is zero on the solutions. This special feature of the lagrangian requires to look for another procedure leading to corresponding conserved quantities. A good candidate seems to be $T^{\mu\nu} = \phi^2 \bar{\zeta}^\mu \bar{\zeta}^\nu$. In fact, we obtain

$$\nabla_\nu T^{\mu\nu} = \bar{\zeta}^\mu \nabla_\nu (\phi^2 \bar{\zeta}^\nu) + \phi^2 \bar{\zeta}^\nu \nabla_\nu \bar{\zeta}^\mu.$$

The first term on the right is equal to $\bar{\zeta}^\mu L_{\bar{\zeta}}\phi^2 = 0$, and the second term is zero because $\bar{\zeta}$ is autoparallel.

It is also easily seen that this lagrangian is $U(1)$-invariant. The corresponding conserved quantity looks like $-\kappa l_o E$ where E is the integral energy, so, after deriving by the speed of light c this quantity becomes $\pm E l_o / c = \pm ET$.

Most of these relations can be easily carried to a lagrangian defined in terms of a 2-form in the following way. Recall the complex structure $\mathcal{J}$ and define a representation ρ of the algebra $\mathbb{C}$ in the algebra $L_{\Lambda^2(\mathbb{R}^4)}$ of linear maps in the 2-forms on $\mathbb{R}^4$ by the relation

$$\rho(\alpha_\varepsilon)=\rho(uI+\varepsilon pJ)\stackrel{\text{def}}{=}u\mathcal{I}+\varepsilon p\mathcal{J},\ \ \mathcal{I}=id_{\Lambda^2(\mathbb{R}^4)},\ \ \alpha_\varepsilon\in\mathbb{C},\ \ \varepsilon=\pm1. \tag{18}$$

Clearly, $\rho(\alpha+\beta)=\rho(\alpha)+\rho(\beta)$, $\rho(\alpha.\beta)=\rho(\alpha)\circ\rho(\beta)$, and if F is an arbitrary 2-form $\rho(\alpha_\varepsilon).F=uF+\varepsilon p\mathcal{J}(F)$. Note that here and further in the text the couple (u,p) may denote the complex number $(uI+pJ)$, as well as the complex-valued function $\alpha=u(x,y,z,\xi)I+p(x,y,z,\xi)J$.

Let's go back now to our $\bar\zeta$-adapted coordinate system and consider the 2-form $F_o=dx\wedge\zeta=\varepsilon dx\wedge dz+dx\wedge\xi$. We obtain

$$\rho(\alpha_\varepsilon).F_o=\varepsilon udx\wedge dz+\varepsilon pdy\wedge dz+udx\wedge d\xi+pdy\wedge d\xi=A\wedge\zeta,$$

$$\rho(J(\alpha_\varepsilon)).F_o=(-\varepsilon p\mathcal{I}+u\mathcal{J}).F_o=$$

$$=-pdx\wedge dz+udy\wedge dz-\varepsilon pdx\wedge d\xi+\varepsilon udy\wedge d\xi=A^*\wedge\zeta=\mathcal{J}(\rho(\alpha_\varepsilon).F_o).$$

In this way we get one-to-one map between the $\mathbb{C}$-valued functions on $\mathbb{R}^4$ and a special subset of 2-forms. All such 2-forms depend on the choice of the 1-form ζ, while the dependence on dx is not essential. Also, they are isotropic, i.e. they have zero invariants.

Moreover, every such 2-form may be considered as a linear map in $\Lambda^2(\mathbb{R}^4)$ through the above correspondence: $\rho(\alpha_\varepsilon).F_o\rightarrow\rho(\alpha_\varepsilon)$. Since together with the zero element of $\Lambda^2(M)$ these 2-forms define a linear space V_ζ, this property suggests to introduce inner product in this linear space by the rule $<F^1_\varepsilon(a,b),F^2_\varepsilon(m,n)>=\frac{1}{6}tr\Big[\rho\big[\alpha_\varepsilon(m,n)\big]\circ\rho\big[\alpha^*_\varepsilon(a,b)\big]\Big]=am+bn$. Hence, every such 2-form acquires a norm. Note that this inner product coincides with the induced by a pseudoeuclidean metric η inner product g on the homology space generated by the corresponding energy-tensor as we mentioned in the preceding section. But, the richer nature of the inner product $<,>$ is manifested in that it generates a Hermitian one in V_ζ with respect to the natural product

$$\lambda_\varepsilon.F_\varepsilon(u,p)=\rho(\lambda_\varepsilon)\circ\rho\big[\alpha_\varepsilon(u,p)\big]=\rho\big[\lambda_\varepsilon.\alpha_\varepsilon(u,p)\big],\ \ \lambda_\varepsilon=const\in\mathbb{C},$$

under the correspondence $F_\varepsilon(u,p)\rightarrow\rho(\alpha_\varepsilon(u,p))$.

Let now F and G be two arbitrary 2-forms. In order to define the lagrangian we consider the Minkowski space-time $M=(\mathbb{R}^4,\eta)$ as a real manifold, where the pseudoeuclidean metric η has signature $(-,-,-,+)$, and make use of the Lie derivative $L_{\bar\zeta}$ with respect to the vector field $\bar\zeta$. Also, $-(k^s)^2=(k^\xi)^2=(l_o)^{-2}$. The lagrangian is given by

$$\mathbb{L}=\eta\left(\kappa l_oL_{\bar\zeta}G+F,F\right)-\eta\left(\kappa l_oL_{\bar\zeta}F-G,G\right)=$$

$$=\frac{1}{2}\left(\kappa l_o\bar\zeta^\sigma\frac{\partial G_{\alpha\beta}}{\partial x^\sigma}+F_{\alpha\beta}\right)F^{\alpha\beta}-\frac{1}{2}\left(\kappa l_o\bar\zeta^\sigma\frac{\partial F_{\alpha\beta}}{\partial x^\sigma}-G_{\alpha\beta}\right)G^{\alpha\beta},\ \ 0<l_o=const,\ \ \kappa=\pm1, \tag{19}$$

where $F^{\alpha\beta}=\eta^{\alpha\mu}\eta^{\beta\nu}F_{\mu\nu}$ and $G^{\alpha\beta}=\eta^{\alpha\mu}\eta^{\beta\nu}G_{\mu\nu}$. Note that this lagrangian is invariant with respect to $(F,G)\rightarrow(G,-F)$, or to $(F,G)\rightarrow(-G,F)$.

The corresponding equations read

$$\kappa l_o\bar\zeta^\sigma\frac{\partial G_{\alpha\beta}}{\partial x^\sigma}+F_{\alpha\beta}=0,\ \ \kappa l_o\bar\zeta^\sigma\frac{\partial F_{\alpha\beta}}{\partial x^\sigma}-G_{\alpha\beta}=0.$$

In coordinate-free form these equations look like

$$\kappa l_oL_{\bar\zeta}G=-F,\quad \kappa l_oL_{\bar\zeta}F=G.$$

Recall now the complex structure $\mathcal{J}$ and assume $G = \mathcal{J}(F)$. Then, treating F and $\mathcal{J}(F)$ as independent (in fact they are linearly independent on the real manifold M) we can write

$$\kappa l_o L_{\bar{\zeta}}\mathcal{J}(F) = -F, \quad \kappa l_o L_{\bar{\zeta}}F = \mathcal{J}(F). \tag{20}$$

Since in our coordinates $\mathcal{J}$ and $\bar{\zeta}$ have constant coefficients, clearly, $L_{\bar{\zeta}}$ and $\mathcal{J}$ commute, so that the second (first) equation is obtained by acting with $\mathcal{J}$ from the left on the first (second) equation, i.e. the above mentioned invariance with respect to the transformations $(F,G) \rightarrow (\pm G, \mp F)$ is reduced to $\mathcal{J}$-invariance. Recalling now the way $\mathcal{J}$ acts, $\mathcal{J}(A\wedge\zeta) = A^*\wedge\zeta$, i.e. the couple (A,ζ) is rotated to the couple (A^*,ζ), we naturally interpret the last equations as realization of the *translational-rotational consistency*: the translational change of F along $\bar{\zeta}$ is proportional to the rotational change of F determined by $\mathcal{J}$, so, roughly speaking, **no $\bar{\zeta}$-translation ($\mathcal{J}$-rotation) is possible without $\mathcal{J}$-rotation ($\bar{\zeta}$-translation)**, and the $\mathcal{J}$-rotation corresponds to l_o translational advancement.

From (20) it follows that on the solutions the lagrangian becomes zero. So, if we try to define the corresponding Hilbert energy-momentum tensor the variation of the volume element with respect to η is not essential. Moreover, the special quadratic dependence of $\mathbb{L}$ on η shows that the variation of $\mathbb{L}$ with respect to η will also become zero on the solutions. Hence, this is another example of the non-universality of the Hilbert method to define appropriate energy-momentum tensor. As for the canonical energy-momentum tensor, it is not symmetric, and its symmetrization is, in some extent, an arbitrary act, therefore, we shall not make use of it.

In order to come to more realistic relations we shall restrict equations (20) on the subset of 2-forms F of the above defined kind, namely, $F = \rho(\alpha_\varepsilon).F_o$. All these 2-forms have zero invariants: $F_{\mu\nu}F^{\mu\nu} = F_{\mu\nu}(\mathcal{J}(F))^{\mu\nu} = 0$, or in coordinate-free way, $F\wedge F = F\wedge\mathcal{J}(F) = 0$. Moreover, the easily verified relations $i(\bar{\zeta})F = i(\bar{\zeta})\mathcal{J}(F) = 0$ show an *intrinsic* connection to $\bar{\zeta}$: it is the only isotropic eigen vector of $F_\mu^\nu = \eta^{\mu\sigma}F_{\nu\sigma}$ and $(\mathcal{J}F)_\mu^\nu = \eta^{\mu\sigma}(\mathcal{J}F)_{\nu\sigma}$.

Substituting $F = \rho(\alpha_\varepsilon).F_o$ in (20) we get the equations

$$\kappa l_o(u_\xi - \varepsilon u_z) = -\varepsilon p, \quad \kappa l_o(p_\xi - \varepsilon p_z) = \varepsilon u, \tag{21}$$

i.e. $\kappa l_o L_{\bar{\zeta}}u = -\varepsilon p$ and $\kappa l_o L_{\bar{\zeta}}p = \varepsilon u$. From these equations we obtain the relations

$$(u^2+p^2)_\xi - \varepsilon(u^2+p^2)_z = \frac{1}{2}L_{\bar{\zeta}}(u^2+p^2) = 0, \quad u(p_\xi - \varepsilon p_z) - p(u_\xi - \varepsilon u_z) = \frac{\varepsilon\kappa}{l_o}(u^2+p^2). \tag{22}$$

The substitution $u = \phi\,\cos(\psi)$, $p = \phi\,\sin(\psi)$ in (22) leads to the equations (16)

$$L_{\bar{\zeta}}\phi = 0, \quad L_{\bar{\zeta}}\psi = \frac{\varepsilon\kappa}{l_o},$$

and to the corresponding solutions (17). Now, relations (11),(13),(14) and (22) suggest to choose $\phi^2\zeta\otimes\bar{\zeta}$ for energy-momentum tensor.

Note that the 2-form $F_o = dx\wedge\zeta$ satisfies the first of these equations since $\phi_{F_o} = 1$ and does NOT satisfy the second equation since $\psi_{F_o} = 0, 2\pi, 4\pi, ...$, so, $L_{\bar{\zeta}}(\psi_{F_o}) = 0$. In view of this further we consider only not-constant $\mathbb{C}$-valued functions.

As for integral characteristic of the intrinsic rotational properties of a solution a good candidate seems to be the integral over $\mathbb{R}^3$ of the restricted to $\mathbb{R}^3$ appropriately interpreted representative of the Godbillon-Vey class as given by relation (15). Making use of the solution ψ_1 as given above and choosing appropriate coefficient, we obtain

$$\beta = \frac{2\pi(l_o)^2}{c}\Big[-\varepsilon\phi^2(\psi_\xi - \varepsilon\psi_z)dx\wedge dy\wedge dz - \phi^2(\psi_\xi - \varepsilon\psi_z)dx\wedge dy\wedge d\xi\Big],$$

so,

$$i^*\beta = 2\pi\frac{l_o}{c}\kappa\phi^2 dx\wedge dy\wedge dz.$$

Since the energy-density is given by the spatially finite function ϕ^2 the corresponding integral of $i^*\beta$ over $\mathbb{R}^3$ gives $\kappa ET = \pm ET$, where E is the integral energy of the solution and $T = 2\pi l_o/c$.

Another also appropriate local representative of the rotational properties of these solutions appears to be any of the two Frobenius 4-forms $\mathbf{d}A \wedge A \wedge A^*$ and $\mathbf{d}A^* \wedge A^* \wedge A$, multiplied by the coefficient $2\pi\varepsilon l_o/c$, so that integrating over the 4-region $(\mathbb{R}^3 \times l_o)$ we get also κET.

The linear character of the equations obtained sets the question if the superposition principle holds. In general, let the parameters κ, ε, l_o of the two solutions be different. Let now $F_1(\kappa_1, \varepsilon_1, l_o^1; u, p)$ and $F_2(\kappa_2, \varepsilon_2, l_o^2; m, n)$ be two solutions along the same direction defined by $\bar\zeta$, and ε of ζ is of course equal to ε_1 for the first solution, and equal to ε_2 for the second solution. We ask now whether the linear combination $c_1F_1 + c_2F_2$ with $c_1 = const, c_2 = const$ will be also a solution along the same direction? In order this to happen the following equations must be consistent:

$$\kappa_1\varepsilon_1 l_o^1 L_{\bar\zeta} u = -\varepsilon_1 p, \quad \kappa_1\varepsilon_1 l_o^1 L_{\bar\zeta} p = \varepsilon_1 u, \quad \kappa_2 l_o^2 L_{\bar\zeta} m = -\varepsilon_2 n, \quad \kappa_2 l_o^2 L_{\bar\zeta} n = \varepsilon_2 m$$

$$\kappa_3\varepsilon_3 l_o^3 L_{\bar\zeta}(c_1u + c_2m) = -\varepsilon_3(c_1p + c_2n), \quad \kappa_3\varepsilon_3 l_o^3 L_{\bar\zeta}(c_1p + c_2n) = \varepsilon_3(c_1u + c_2m),$$

where ε_3 is equal ε_1, or to ε_2. The corresponding consistency condition looks as follows:

$$\kappa_3\varepsilon_3 l_o^3 = \frac{c_1p + c_2n}{\frac{\varepsilon_1\kappa_1c_1}{l_o^1}p + \frac{\varepsilon_2\kappa_2c_2}{l_o^2}n} = \frac{c_1u + c_2m}{\frac{\varepsilon_1\kappa_1c_1}{l_o^1}u + \frac{\varepsilon_2\kappa_2c_2}{l_o^2}m}.$$

For example, the relations $\varepsilon_3\kappa_3 l_o^3 = \varepsilon_2\kappa_2 l_o^2 = \varepsilon_1\kappa_1 l_o^1$ are sufficient for this superposition to be a solution. This means that if the two solutions propagate translationally for example from $-\infty$ to $+\infty$, i.e. $\varepsilon_1 = \varepsilon_2 = -1$, if the rotational orientations coincide, i.e. $\kappa_1 = \kappa_2$, and if the spatial periodicity parameters are equal: $l_o^1 = l_o^2 = l_o^3$, the sum $(c_1F_1 + c_2F_2)$ gives a solution. In general, however, the combination $c_1F_1 + c_2F_2$ will not be a solution.

On the other hand, we can introduce a multiplicative structure in the solutions of the kind $\rho(\alpha_\varepsilon).F_o$. In fact, if $F_1(\kappa_1) = \rho(\alpha^1(\varepsilon_1)).F_o$ and $F_2(\kappa_2) = \rho(\alpha^2(\varepsilon_2)).F_o$ are two such solutions we define their product $F = F_1.F_2$ by

$$F = F_1.F_2 = \rho(\alpha^1.\alpha^2).F_o.$$

Clearly, the amplitude of F is a product of the amplitudes of F_1 and F_2: $\phi_F = \phi_{F_1}.\phi_{F_2}$ and the phase of F is the sum of the phases of F_1 and F_2: $\psi_F = \psi_{F_1} + \psi_{F_2}$. Now, F will be a solution only if

$$\varepsilon_1 = \varepsilon_2 = \varepsilon_F, \quad \frac{\kappa_F}{l_o^F} = \frac{\kappa_1 l_o^2 + \kappa_2 l_o^1}{l_o^1.l_o^2}.$$

From the first of these relations it follows that in order F to be a solution, F_1 and F_2 must NOT move against each other, and then the product-solution shall follow the same translational direction. However, it is allowed F_1 and F_2 to have different rotational orientations, i.e. $\kappa_1 \neq \kappa_2$, then the product-solution will have rotational orientation $\kappa_F = sign(\kappa_1 l_o^2 + \kappa_2 l_o^1)$. Clearly, every subset of solutions with the same $(\varepsilon, \kappa, l_o)$ form a group with neutral element $F_o = \rho(I).F_o$, and $F_\alpha^{-1} = \rho(\alpha^{-1}).F_o$, moreover, the multiplicative group of $\mathbb{C}$ acts on these solutions: $(\alpha_\varepsilon, F_\varepsilon) \rightarrow \rho(\alpha_\varepsilon).F_\varepsilon$.

5 Discussion and Conclusion

The main feature of individual objects of classical mechanics is that they admit the approximation *material point*, i.e. the physical situations considered in the frame of classical mechanics do NOT admit these objects to be created and destroyed, they just change their state of motion

under external influences, so, as theoretical entities, they are *uncreatable* and *indestructible*. In other words, their structure is neglected, they are considered as structureless.

On the contrary, photons appeared in physics as *creatable* and *destructible* objects, i.e. objects with structure, hence, a new kind of theory describing their behavior in accordance with their structure was needed. Although Einstein's interpretation of Planck's formula concerning light quanta in view of the relativity principle, the later developed quantum mechanics continued to consider the point-like approximation of microobjects as admissible and consistent in the context of some probabilistic interpretation. Moreover, a general understanding of the physical situation on the base of some definite *uncertainty/complementarity relations* was formulated. Quantum electrodynamics also has not presented so far a realistic spatially finite and time-stable model of individual photons with dynamical structure.

The two basic features of our approach in this paper are the assumptions for continuous spatially finite structure, and for available consistent translational-rotational dynamical structure of photons. Hence, photons *propagate*, they do not move. The propagation has two components: *translational* and *rotational*. The translational component is along isotropic straight lines in $(\mathbb{R}^4, \eta)$. The rotational component of propagation follows the special rotational properties of photon's spatial structure. This dual nature of photons demonstrates itself according to the rule: **no translation (rotation) is possible without rotation (translation)** (equation (20)).

While the translational component of propagation is easily accounted through the (arbitrary chosen in general) isotropic autoparallel vector field $\bar{\zeta}$, the rotational component of propagation was introduced in our model making use of two things: the non-integrability properties of the induced by $\bar{\zeta}$ two 2-dimensional differential/Pfaff systems, and the complex structure $\mathcal{J}$. This approach brought the following important consequences:

1. It automatically led to the required translational-rotational consistency.

2. The rotational properties of the solutions have *invariant* and *intrinsic* for the object nature, they are transversal to $\bar{\zeta}$, they are in accordance with the action of the complex structure $\mathcal{J}$ on the 2-forms G and G^*, and they allow the characteristics *amplitude* ϕ and *phase* ψ of a solution to be correctly introduced. Moreover, the rotation is of periodical nature, and a helical spatial structure along $\bar{\zeta}$ is allowed, so, the corresponding rotational properties differ from those in the case of rotation of a solid as a whole around a point or axis.

3. The translational properties of a solution are carried by the amplitude ϕ : $L_{\bar{\zeta}}\phi = 0$, and the rotational ones - by the phase ψ: $L_{\bar{\zeta}}\psi \neq 0$.

4. The curvature, considered as a measure of the available non-integrability, is non-zero only if the phase ψ is NOT a running wave along $\bar{\zeta}$: $L_{\bar{\zeta}}\psi \neq 0$, hence, curvature means rotation and vice versa.

5. Quantitatively, the curvature is *obtained* to be proportional to the energy-density ϕ^2 (and to the change of the phase ψ along $\bar{\zeta}$) of a solution. We recall that in General Relativity a proportionality relation between the energy-density of non-gravitational fields and the corresponding (contracted) riemannian curvature is *postulated*, while here it is *obtained* in the most general sense of the concept of curvature, namely, as a measure of Frobenius non-integrability.

6. A natural *integral* measure of the rotational properties of a solution appears to be the product ET, i.e. the action for one period $T = 2\pi l_o/c$, which is in accordance with the Planck formula $ET = h$. It seems remarkable that this quantity may be obtained also through the spatial restriction of the available and correspondingly interpreted Godbillon-Vey class.

Together with the allowed finite nature of the solutions these properties suggest the following understanding of the photons' time-stable dynamical structure: **photons MUST always propagate in a translational-rotational manner as fast as needed in order to "survive", i.e. to overcome the instability (the destroying tendencies), generated by the available non-integrability**. In other words, every free photon has to be able to supply immediately itself with those existence needs that are constantly put under the non-integrability

destroying influence. Some initial steps to understand quantitatively this "smart" nature of photons in the terms used in the paper could be the following.

Recalling the two 2-forms $G = A \wedge \zeta$ and $G^* = A^* \wedge \zeta$ and relations (11)-(12) we see that when $L_{\bar{\zeta}}\phi^2 = 0$ then G and G^* keep the energy-momentum carried by each of them: $i(\bar{G})\mathbf{d}G = i(\bar{G^*})\mathbf{d}G^* = 0$. On the other hand the relation $i(\bar{G})\mathbf{d}G^* = -i(\bar{G^*})\mathbf{d}G = \mathbf{R}.\zeta$ may be physically interpreted in two ways. FIRST, differentially, G transfers to G^* so much energy-momentum as G^* transfers back to G, which goes along with the previous relations stating that G and G^* keep their energy-momentum densities. Each of these two quantities $i(\bar{G})\mathbf{d}G^*$ and $i(\bar{G^*})\mathbf{d}G$ is equal (up to a sign) to $\mathbf{R}.\zeta$, so, such mutual exchange of energy-momentum is possible only if the non-integrability of each of the two Pfaff 2-dimensional systems (A, ζ) and (A^*, ζ) is present and measured by the same non-zero quantity, i.e. when the curvature $\mathbf{R}$ is NOT zero. Since the curvature implies outside/inside directed flow, this suggests the SECOND interpretation: the energy-momentum that photons lose differentially in *whatever way* by means of G is differentially supplied by means of G^*, and vice versa. We could say that every photon has two functioning subsystems, G and G^*, such, that the energy-momentum loss through the subsystem G generated by the nonintegrability of (A, ζ), is gained (or supplied) back by the subsystem G^*, and vice versa, and in doing this photons make use of the corresponding rotational component of propagation supported by appropriate spatial structure. All this is mathematically guaranteed by the isotropic character of G and G^*, i.e. by the zero values of the two invariants $G_{\mu\nu}G^{\mu\nu} = G^*_{\mu\nu}G^{\mu\nu} = 0$, and by making use of the complex structure $\mathcal{J}$ as a rotation generating operator.

Let's now try to express this dually consistent dynamical nature of photons by *one* object which satisfies *one* relation. We are going to consider G and G^* as two vector components of a $(\mathbb{R}^2, J)$-valued 2-form, namely, $\Omega = G \otimes I + G^* \otimes J$. Applying the exterior derivative we get $\mathbf{d}\Omega = \mathbf{d}G \otimes I + \mathbf{d}G^* \otimes J$. Consider now the $(\mathbb{R}^2, J)$-valued 2-vector $\bar{\Omega} = \bar{G} \otimes I + \bar{G^*} \otimes J$. The aim we pursue will be achieved through defining the object ($\vee$ is the symmetrized tensor product)

$$(\vee, i)(\bar{\Omega}, \mathbf{d}\Omega) \stackrel{\text{def}}{=} i(\bar{G})\mathbf{d}G \otimes I \vee I + i(\bar{G^*})\mathbf{d}G^* \otimes J \vee J + \Big[i(\bar{G})\mathbf{d}G^* + i(\bar{G^*})\mathbf{d}G\Big] \otimes I \vee J$$

and put it equal to zero: $(\vee, i)(\bar{\Omega}, \mathbf{d}\Omega) = 0$.

Finally we note that this last relation $(\vee, i)(\bar{\Omega}, \mathbf{d}\Omega) = 0$ represents the dynamical equations of the vacuum Extended Electrodynamics [24]. In particular, this equation contains all solutions to the Maxwell vacuum equations, and the solutions obtained in this paper are a special part of the full subset of nonlinear solutions to these nonlinear equations.

In conclusion, spatially finite field models of photons can be built in terms of complex valued functions and in terms of 2-forms on Minkowski space-time, and these two approaches can be related. In these both cases substantial role play the complex structures J and $\mathcal{J}$ as rotation generating operators, carrying information in this way about the spin properties of photons. This moment throws some light on the long standing and nonanswered question why quantum theory works with $(\mathbb{R}^2, J)$-valued objects and not just with $\mathbb{R}^2$-valued ones: may be just the spin properties of micro-objects, induced by their dynamical structure, are, at least partially, accounted for in this way.

The photons' longitudinal dimensions and rotational orientations can be determined by the constant parameter combination κl_o, and their transversal dimensions depend on the energy-density. The time-stability is guaranteed, on one hand, by the internal energy-momentum exchange between the two non-integrable 2-dimensional differential/Pfaff systems, and on the other hand, by a dynamical harmony with the outside world. The models obtained are consistent with the present day knowledge about the propagational behavior and the integral energy-momentum and spin characteristics of photons.

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